The Impact of a Finite Bankroll on an Even-Money Game

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Calculating the average cost of playing a table game is usually equated to the product of the house edge, the average wager and the number of hands played. A linear amount that is static, providing that the player has enough financial resources to play for the specified duration. However, in the twilight of a healthy wallet, the probability of survival creeps into the equation, creating the associated cost of not having enough capital to guarantee play for the intended length of the session.

To show the basic concept, we can say that the cost of playing a table game is:
n * W * E = C, Where **n** = The intended number of plays, **W** = Average wager, **E** = House advantage, and finally **C** = The cost of playing. This holds true providing that (**W** * **n**) < **B**, where **B** = the player's Bankroll. However, when (**W** * **n**) > **B** the formula above will fall short, unless some modifications are made.

Properties of R

The scenario of Akio Kashiwagi at the Trump Plaza provides an interesting situation to consider while delving into this portion. In their paper entitled "Dealing to the Premium Player: Casino Marketing and Management Strategies to Cope with High Risk Situations", Bill Eadington and Nigel Kent-Lemon refer to an article printed in the Wall Street Journal[1], in which the story is told of Kashiwagi's agreement with Trump Plaza to play a game of "freeze-out". The terms of play stipulated that Kashiwagi wager $200,000 per hand on the game of Baccarat until he either won or lost 12,000,000, after which point the game would be terminated.

Eadington/Kent-Lemon used a description of "gambler's ruin"[2] to show the probabilities of the advantaged side ultimately losing or winning a game as $(q/p)^u$ for losing and $1 - (q/p)^u$ for winning, where p = the single trial probability of winning; q = 1 – p, which is the single trial probability of losing; p > q; and finally u = the number of betting units that, once lost, will end the game.

Although the standard game of Baccarat has three betting options, it is inferred that no Tie bets were taken. Also, given that the game was intended to be played until the conclusion, it is further inferred that the probabilities used for both the Player and Banker bets, exclude the possibilities of Ties. For this, 0.493 is the probability that Player bet will occur on a single trial and 0.507 is the probability that a Banker

[1] Robert Johnson – "Tale of a Whale: A Gambler Wins, Loses Millions – And Duels Trump" (1990)

[2] Peter Griffin – "Extra Stuff: Gambling Ramblings"

bet will. To compensate for the 5% commission charge, which results when the Banker bet is played, I assume the following adjustments were made: Banker = 0.494 & Player = 0.506. The 12,000,000 bankroll and the $200,000 wagers translates into a 60 unit swing. With this information Eadington/Kent-Lemon approximated the probability that the casino (the advantaged side) would eventually lose the game to be $(0.493/0.507)^{60}$ = 18.6% if the player bets Player and $(0.494/0.506)^{60}$ = 23.7% if the player bets Banker.

Although it is unclear as to how he perceived his chances, Kashiwagi accepted the challenge and subsequently went on to play for a total of 70 hours over 6 days. At this point the game was prematurely terminated and Kashiwagi found himself $9,400,000 (47 Units) in the hole. According to the New York Times, both sides denied responsibility for ending the game. Given that this game has since taken on mythological status, one can only speculate as to what actually brought an end to the session. Nevertheless, I will offer one possibility.

Assuming the average person could actually stomach a steady diet of “marinated monkey meat” for six straight days - which Kashiwagi was reported to have eaten , I would suggest that the length of play may have been one of the predominant factors resulting in the deal being terminated. Seventy hours over 6 days averages out to be just under 12 hours of play per day. This could be a bit too much Baccarat, even for a hardcore player. That being said, what comes to light is the limitation of using the previous calculations: $(q/p)^u$ & $1 - (q/p)^u$. That is, they assume that the player (the disadvantaged side = q) will play as long as it is required to reach the specified goal of winning a predefined amount of units (u) or losing to a point where recovery is not possible[3]. The length of time the average players plays, much like Kashiwagi, is unpredictable; particularly when their bankrolls are unknown.

To look at the chances of this game being resolved at any intermediate time interval requires us to do some statistical calculations. In Kashiwagi’s case, the first would be the variance of baccarat, which for the Banker wager is $(0.95\text{-}E)^2 * 0.4586 + (-1\text{-}E)^2 * 0.4462 + E^2 * 0.0952 = 0.860$ and the Player wager is $(-1\text{-}E)^2 * 0.4586 + (1\text{-}E)^2 * 0.4462 + E^2 * 0.0952 = 0.905$. Note that E for the Banker bet is –0.0106 and E for the Player bet it is -0.0124. By taking the mean of the deviations (σ) = $(\sqrt{(0.86)} + \sqrt{(0.905)}) / 2 = 0.94$ and multiplying it by the root of the number of hands played, we can obtain the standard deviation for 5,000 hands: $\sqrt{(5000)} * 0.94 = 66.47$.

Also, by taking the average of the two E’s and multiplying it by the number of hands, we can obtain the mean number of units won: (-0.0106 + -0.0124) / 2 * 5000 = -57.5. These two numbers give us the components that are required to standardize the distribution of win probabilities: $z = f(x) = (x – (-57.5)) / 66.47$.

Calculating the distribution requires the use of the standard normal cumulative distribution function:

[3] This in fact is a one-sided barrier calculation. The Kashiwagi scenario was a two-sided barrier problem, as both Kashiwagi and Trump had the barrier of ± 60 units. In his book, Griffin shows that the probability of doubling before ruin as $q^u / (q^u + p^u)$. This calculation applies to the Kashiwagi/ Trump agreement as both barriers were equal. Although this paper only delves into the one-sided barrier effect, the distributions can be further modified to solve two-sided barrier problems.

$$F(z) = \int_{-\infty}^{z} 1/\sqrt{(2\pi)} * e^{-(1/2)x^2} dx$$

Range			z Range (z = f(x) = (x – (-57.5)) / 66.47)			Culmulative	Win Probabilities		
								Monkey	
Lower		Upper	Lower (zl)		Upper (zu)	F(zu) - F(zl)	Trump	Meat	Kashiwagi
Below	to	-120	-Infinity	to	-0.940273808	17.35%	17.35%	0.00%	0.00%
-120	to	-100	-0.940274	to	-0.639386189	8.77%	8.77%	0.00%	0.00%
-100	to	-80	-0.639386	to	-0.338498571	10.62%	10.62%	0.00%	0.00%
-80	to	-60	-0.338499	to	-0.037610952	11.75%	11.75%	0.00%	0.00%
-60	to	-40	-0.037611	to	0.263276666	11.88%	0.00%	11.88%	0.00%
-40	to	-20	0.263277	to	0.564164285	10.98%	0.00%	10.98%	0.00%
-20	to	0	0.564164	to	0.865051903	9.28%	0.00%	9.28%	0.00%
0	to	20	0.865052	to	1.165939522	7.17%	0.00%	7.17%	0.00%
20	to	40	1.16594	to	1.46682714	5.06%	0.00%	5.06%	0.00%
40	to	60	1.466827	to	1.767714759	3.27%	0.00%	3.27%	0.00%
60	to	80	1.767715	to	2.068602377	1.93%	0.00%	0.00%	1.93%
80	to	100	2.068602	to	2.369489995	1.04%	0.00%	0.00%	1.04%
100	to	Above	2.36949	to	Infinity	0.89%	0.00%	0.00%	0.89%
						100.00%	**48.50%**	**47.64%**	**3.86%**

Table A (Note: no correction for continuity)

After 5000 wagers, this table shows that Kashiwagi had almost an even chance of losing or continuing to feast on marinated monkey meat; which in fact , after landing in the 56th percentile, he should have. However, despite the useful information obtained from the distribution in the this table, there is an additional factor that should be considered.

Note that the probability of a player landing in the range of –60 to –40 units after 5000 hands of baccarat is shown to be 11.88%. Now consider the scenario of a player winning 2,475 times in a row, preceded by 2,525 straight losses. This lands him in the aforementioned range, with a net of –50 units. An unlikely scenario, but a possibility nevertheless; and one that is factored in the 11.88% despite the fact that a player with a 60 unit bankroll could not withstand 2,525 straight losses.

I'll demonstrate this with the Illustration below, which intentionally resembles Pascal's triangle. Imagine a game where a man has to hop down to the bottom row from the top square. The man is not allowed to hop upwards, but can only hop to the row below, hopping one row at a time. Each time he hops down to our left he is penalized one unit and each time down to our right he earns one unit. If the man only has **3** units, he would be out of units if he landed on any of the dark shaded squares. The distribution we calculated earlier would have only considered the three shaded squares on the bottom row as ruins. Therefore, if the man followed the path shown, he would have been counted as being one unit ahead instead of being broke.

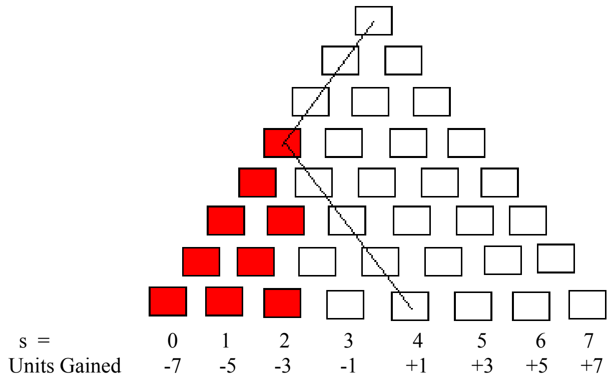

Illustration A

It is interesting to note that, for a game where the units that are won and lost are of equal value and where p + q = 1, any odd number of betting units can only be depleted after an odd number of trials. Equally, the same is true for an even number of units and trials. For this reason, the following distribution only works in such cases.

Binomial Ruin Distribution

Let C(n,s) = n! / s!(n-s)! where s = the number of wins. The ruin point (r) at the end of n trials can be shown as r = s = (n – u) / 2, where u represents the number of units in a bankroll. For the case shown above, the ruin point at the end of 7 trials, given a bankroll of 3 units, is r = s = (7 –3) / 2 = 2. Counting the first shaded square as zero, we get C(7,0), C(7,1) & the ruin point C(7,2) representing the last three shaded squares at the bottom of the triangle. For the binomial distribution, where p = 0.49425 and q = 0.50575, the cumulative probability of all three shaded squares equal:

$$\sum_{X=0}^{r} C(7,x) * 0.49425^{x} * 0.50575^{7-x} = 0.2361$$

To add the probability of ruin before the end, the calculation must be extended to the following:

$$\mathbf{R} = \sum_{X=0}^{r} C(n,x) * p^{x} * q^{n-x} + \sum_{X=0}^{r-1} C(n,x) * p^{2r-x} * q^{u+x}$$

Doing this, we obtain an overall ruin probability of 29.91% instead of the 23.61% calculated at the end. The latter part of this calculation has an impact on the distribution in the range of r + 1 to 2r, or in this case 3 to 4, and can be described with the following:

$$(C(n,x) - C(n, 2r - x)) * p^{x} * q^{n-x}$$

This results in 3 being 22.118% and 4 being 26.247%. The remainder, 5 to 7, distributes per usual.

$C(n,x) * p^{x} * q^{n-x}$

Outcome	Distribution
Ruin	0.299118205
3	0.221177849
4	0.262466152
5	0.158425274
6	0.05160764
7	0.00720488
	1.00000000

Table 1

What's interesting about this calculation is that it not only shows the ruin effect, but it also shows the effect that ruin has on the remainder of the distribution**.** To test the validity of these calculations I ran a simulation of 102,816 sequences, which yielded the following comparison:

	Simulation	
Result	Trials	Frequencies
Ruin	30553	0.297161920
-1	22859	0.222329209
1	27177	0.264326564
3	16064	0.156240274
5	5406	0.052579365
7	757	0.007362667
	102816	1.000000000

Table 2

Although this method of calculating the ruin distribution is highly accurate, the large numbers tend to make things a little cumbersome; particularly when the binomial function on an excel spreadsheet tends to give out with just over a thousand trials. Luckily, the binomial lends itself to continuous approximations.

Don Schlesinger & Continuous Approximations to the Binomial Ruin Distribution

Although Alan Krigman's paper states that it is a "work in process", he describes the survival criterion[4]. This is based on an observation made by Don Schlesinger, who noted the importance of considering both the "end-point" and "barrier" effects. To reiterate the distinction between the two, the "end-point" refers to the probability of a player's bankroll **u** (in units) being at or below 0 after **n** trials, and the "barrier" refers to the player's bankroll being at or below 0 during any point before **n** trials. In his paper, Krigman shows Schlesinger's calculation for the probability of a player going broke on or before **n** trials as:

$$\mathbf{R} = F(z_1) + F(z_2)\, e^{-2Eu / V}$$

Where $\mathbf{F(z_1)}$ represents the "endpoint" effect, and the remainder of the equation represents the "barrier" effect. **V** represents the Variance of the game (σ^2) and **E** is the house edge.

$\mathbf{z_1}$ and $\mathbf{z_2}$ are standardized with the following equations:

$$\mathbf{z_1} = (-u - En)/\sqrt{(Vn)} \quad \& \quad \mathbf{z_2} = (-u + En)/\sqrt{(Vn)}$$

As a comparison, the following shows that the two are fairly close. Also, I would extrapolate that if $p > 0.5$ in the Binomial Ruin, or if $E > 0$ in Schlesinger's calculation, then as n approaches ∞ both bring us full circle to approximate the ruin described in the Eadington / Kent-Lemon paper:

$$\mathbf{R} = (q/p)^u$$

p = 0.50575, q = 0.49425, u = 60, E = 0.0115

n	Binomial Ruin	Schlesinger Formula	(q / p) ^ u
1,000	2.767%	2.762%	25.156%
5,000	17.054%	17.053%	25.156%
10,000	21.839%	21.839%	25.156%
20,000	24.288%	24.288%	25.156%
50,000	25.117%	25.116%	25.156%

Table 3

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[4] Alan Krigman – "Why What You Got Ain't What You Expected: Edge, Volatility, & Risk of Ruin in Gambling"

Borrowing from the Schlesinger formula and using the Normal Approximation to the Binomial[†], as well as the Gamma Function[††], we can restate the Binomial Ruin Distribution in the following manner:

$F((r+0.5-np)/\sqrt{(npq)}) + F((r-0.5-nq)/\sqrt{(npq)})\ e^{2u(q-p) / (1-(p-q)^{\wedge}2)}$	for ruin
$(1- e^{\ln(\Gamma(x+1)) + \ln(\Gamma(n-x+1)) - \ln(\Gamma(2r-x+1)) - \ln(\Gamma(u+x+1))}) * (F(z_a) - F(z_b))$	where $r < x < 2r + 1$
$F(z_a) - F(z_b)$	where $x > 2r$

[†] $F(z_a) - F(z_b)$ is the basic Normal Approximation to the binomial with adjustment for continuity:

$F((x+0.5-np)/ \sqrt{(npq)}) - F((x-0.5-np)/ \sqrt{(npq)})$

[††] Although the use of logarithms in the exponent appears redundant, it becomes practical when utilizing spreadsheets. Microsoft Excel, for instance, cannot compute anything over 170!or $\Gamma(169)$ and the use of a Normal Approximation is too tedious. The Excel function =gammaln(x) readily computes $\ln(\Gamma(x))$. These create lower values, which, after subtraction, expands the range for the higher values of x & n.

A Modification of n * W * E = C

At the beginning of this document, I stated that the cost of playing a table game (n * W * E = C) had to be modified when (W * n) > B, where W = the average wager and B = the player's bankroll. For the effect of ruin, calculating the expected gain in units and subsequently multiplying it by the average wager, gives us the modified amount. This can be carried out with the following:

$$\mathbf{Eu_1} = \left(\sum_{x=0}^{r} C(n,x) * p^{x} * q^{n-x} + \sum_{x=0}^{r-1} C(n,x) * p^{2r-x} * q^{u+x} \right) * -u$$

$$\mathbf{Eu_2} = \sum_{x=r+1}^{2r} (C(n,x) - C(n, 2r - x)) * p^{x} * q^{n-x} * (2x - n)$$

$$\mathbf{Eu_3} = \sum_{x=2r+1}^{n} C(n,x) * p^{x} * q^{n-x} * (2x - n)$$

$$\mathbf{C = W * (Eu_1 + Eu_2 + Eu_3)}$$

For the scenario we looked at earlier (Illustration A & Table 1), we'll start by calculating the constant r = (n – u) / 2; where n = 7, u = 3. If p = 0.49425 and q = 1- p, we can show that Eu_1= -0.8974, Eu_2 = 0.0413, & Eu_3 = 0.7837. So, if a player wagered $25 on each trial, his expected value would be:

C = $25 * (-0.8974 + 0.0413 + 0.7837) = -$1.81 instead of the C = 7 * $25 * (2p – 1) = -$2.01 calculated when (W * n) < B.

Properties of n (with Implications on "Turnover" and Determining a Player's Worth)

It can be shown that by dividing the expected gain in units (Eu_1+ Eu_2+ Eu_3) by the single trial expectation (2p – 1), we get the mean number of trials played. For example, the previously calculated distribution shows a gain of -0.8974 + 0.0413 + 0.7837 = -0.0724. If the single trial expectation is 2 * 0.49425 – 1 = -0.0115, then the average number of trials that will be played is –0.0724 / -0.0115 = 6.29. This means that, in this case, an average of 0.71 trials (or 10%) are lost due to ruin. With this, we can still apply the calculation of n * W * E providing that we make the adjustment to the value of n:

6.29 * $25 * -0.0115 = -$1.81

By utilizing the continuous approximations for higher values, we can determine not only the average length of play, but also the worth of the player. That is, of course, if we have knowledge of both the player's resources (u) and his intended length of play (n). This brings up an interesting fact, which is that a player who intends to play n hands is not worth n hands if his bankroll has limitations: (W * n) > B. This has some implications on comping practices that are based on observations of the player. For instance, if a player with a 20 unit bankroll of $500 is playing a game with the same single trial probabilities we used earlier, and plays for a period of time in which he survives to wager on 700 trials, then we will estimate his value as 700 * $25 * 0.0115 = $201.25. In fact the player's true value is only $138.05, as with a 20 unit bankroll a player will, on average, only see 68.6% of 700 hands. For this reason, whether or not a player is winning should be considered when assuming the player's worth.